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DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

265. Proposed by G. W. GREENWOOD, M. A., Dunbar, Pa.

Obtain the reduced cubic $4\theta^3 - I\theta + J = 0$ of the biquadratic $ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0$.

I. Solution by the PROPOSER.

Assume $ax^4 + 4bx^3 + 6cx^2 + 4dx + e \equiv a(x^2 + 2mx + p)(x^2 + 2nx + q)$.

$\therefore a(m+n) = 2b$; $a(4mn + p + q) = 6c$; $a(mq + np) = 2d$; $apq = e$.

Let $amn = c - \theta$; then $a(p + q) = 2c + 4\theta$.

Substituting in the identity

$$\begin{vmatrix} 1 & 1 & 0 \\ m & n & 0 \\ p & q & 0 \end{vmatrix} \times \begin{vmatrix} 1 & 1 & 0 \\ n & m & 0 \\ q & p & 0 \end{vmatrix} \equiv \begin{vmatrix} 2 & m+n & p+q \\ m+n & 2mn & mq+np \\ p+q & mq+np & 2pq \end{vmatrix} \equiv 0$$

we obtain $\frac{8}{a^3} \begin{vmatrix} a & b & c+2\theta \\ b & c-\theta & d \\ c+2\theta & d & e \end{vmatrix} = 0$; i. e., $4\theta^3 - I\theta + J = 0$.

II. Solution by L. E. NEWCOMB, Los Gatos, Cal.

$$ax^4 + 4bx^3 + 6cx^2 + 4dx + e \equiv x^4 + \frac{4bx^3}{a} + \frac{6cx^2}{a} + \frac{4dx}{a} + \frac{e}{a} = 0 \dots (1).$$

Let $x = y/a$; then (1) becomes $y^4 + 4by^3 + 6acy^2 + 4a^2dy + a^3e = 0 \dots (2)$.

$y^4 + 4by^3 + 6acy^2 + 4a^2dy + a^3e + (ay + \beta)^2 = (y^2 + 2by + \lambda)^2$, by Ferrari's solution; whence, by equating coefficients of like powers of y , and eliminating α and β , $\lambda^3 - 3ac\lambda^2 + a^4(4bd - ae)\lambda - a^3(2d^2 + 2b^2e - 3ace) = 0$. Let $4\frac{1}{3}\theta + ae = \lambda$, then

$$4\theta^3 - 4\frac{1}{3}a^2(3c^2 + ac - 4bd)\theta + a^3[c(4bd - ac - 2c^2) - (2cd^2 + 2b^2e - 3ace)] = 0 \dots (3),$$

which is of the required form.

266. Proposed by L. E. NEWCOMB, Los Gatos, Calif.

Find the n th term and the sum of n terms of the series $1 + 3 + 7 + 17 + \dots$

I. Solution by THEODORE L. DeLAND, Treasury Department, Washington, D. C.

Let $u_1, u_2, u_3, u_4, \dots, u_n$ be the terms of the series, in the Calculus of Finite Differences, and let $\Delta u_1, \Delta^2 u_1, \Delta^3 u_1$, and $\Delta^4 u_1$ be the symbols for the different orders of differences.

The differences and the terms of this special series may be arranged as follows:

$$\begin{array}{cccccccc}
 u_1 & = & 1 & & 3 & & 7 & & 17 & & . & & . & & . \\
 \Delta u_1 & = & . & 2 & & 4 & & 10 & & . & & . & & . \\
 \Delta^2 u_1 & = & . & . & 2 & & 6 & & . & & . & & . \\
 \Delta^3 u_1 & = & . & . & . & 4 & & . & & . & & . & & .
 \end{array}$$

This shows that the third difference of u_1 or $\Delta^3 u_1$, is constant; and that, therefore, the fourth, and all higher differences, in this series, must vanish.

We have always, $u_1 = u_1$; and then we have for the next term, $u_2 = u_1 + \Delta u_1 = u_1(1 + \Delta)$; and then, $u_3 = u_1 + \Delta u_1 + \Delta(u_1 + \Delta u_1) = u_1 + \Delta u_1 + \Delta u_1 + \Delta^2 u_1 = u_1 + 2\Delta u_1 + \Delta^2 u_1 = u_1(1 + \Delta)^2$; and so on for higher orders; and in which Δ may be considered, first, as a symbol of operation, and second, as a symbol of quantity.

The symbols and operations may now be exhibited as follows, as they conform to the law of the Binomial Theorem:

$$\begin{array}{l}
 u_1 = u_1 \\
 u_2 = u_1(1 + \Delta) \\
 u_3 = u_1(1 + \Delta)^2 \\
 u_4 = u_1(1 + \Delta)^3 \\
 . \quad . \quad . \quad . \quad . \\
 u_n = u_1(1 + \Delta)^{n-1}
 \end{array}$$

Expand the term for u_n and we have for its value:

$$u_n = u_1 \left[1 + (n-1)\Delta + \frac{(n-1)(n-2)}{2} \Delta^2 + \frac{(n-1)(n-2)(n-3)}{2 \times 3} \Delta^3 + \dots \right] \quad (A).$$

Let S_n = the value of the sum of n terms and we have:

$$S_n = u_1 + u_2 + u_3 + u_4 + \dots + u_n.$$

We also have for the second member:

$$S_n = u_1 [1 + (1 + \Delta) + (1 + \Delta)^2 + (1 + \Delta)^3 + (1 + \Delta)^4 + \dots + (1 + \Delta)^{n-1}] \dots (B).$$

Sum (B) and we have:

$$S_n = \frac{u_1 [(1 + \Delta)^n - 1]}{\Delta}.$$

Expand the term, $(1 + \Delta)^n$, subtract 1, and divide by Δ , and we have:

$$\begin{aligned}
 S_n = u_1 \left[n + \frac{n(n-1)}{2} \Delta + \frac{n(n-1)(n-2)}{3!} \Delta^2 + \frac{n(n-1)(n-2)(n-3)}{4!} \Delta^3 \right. \\
 \left. + \dots \right] \dots (C).
 \end{aligned}$$

In (A) and (C) remove the brackets so as to unite the symbols of operation and the symbols of quantity and we have:

$$u_n = u_1 + (n-1)\Delta u_1 + \frac{(n-1)(n-2)}{2!}\Delta^2 u_1 + \frac{(n-1)(n-2)(n-3)}{3!}\Delta^3 u_1 + \dots (D);$$

and

$$S_n = nu_1 + \frac{n(n-1)}{2}\Delta u_1 + \frac{n(n-1)(n-2)}{3!}\Delta^2 u_1 + \frac{n(n-1)(n-2)(n-3)}{4!}\Delta^3 u_1 + \dots (E).$$

In (D) and (E) substitute the values, $u_1=1$, $\Delta u_1=2$, $\Delta^2 u_1=2$, and $\Delta^3 u_1=4$, from the problem and its differences, and we have, after reduction:

$$u_n = \frac{1}{3}[2n^3 - 9n^2 + 19n - 9] \dots (F); \text{ and } \\ S_n = \frac{1}{6}[n^4 - 4n^3 + 11n^2 - 2n] \dots (G).$$

Equations (F) and (G) are true for all values of n for the special series under consideration. When $n=4$, $u_n=u_4=17$, and $S_n=S_4=28$, as may be seen by inspecting the series in the problem.

But equations (D) and (E) are perfectly general when the series follows any regular law of progression; as we have to know, only, the value of the leading term, and the leading differences up to the difference that vanishes, to find the value of any term in a series and the sum of that series.

II. Solution by L. E. NEWCOMB, Los Gatos, Cal., and G. W. GREENWOOD, M. A., Dunbar, Pa.

Let $S \equiv u_1 + u_2 x + u_3 x^2 + \dots$ where $u_1=1$, $u_2=3$, $u_3=7$, and, in general, $u_n = 2u_{n-1} + u_{n-2}$, x , of course, being less than unity, numerically.

$$\therefore (1-2x-x^2)S = u_1 + (u_2 - 2u_1)x; \text{ i. e.,}$$

$$S = \frac{1+x}{1-2x-x^2} = \frac{A}{1-\alpha x} + \frac{B}{1-\beta x}$$

where $\alpha=1+\sqrt{2}$, $\beta=1-\sqrt{2}$, $A=\frac{1}{2}(1+\sqrt{2})$, $B=\frac{1}{2}(1-\sqrt{2})$.

$$\therefore u_n = A\alpha^{n-1} + B\beta^{n-1} = \frac{1}{2}[(1+\sqrt{2})^n + (1-\sqrt{2})^n].$$

Let $S_n = u_1 + u_2 + \dots + u_n$; $S_n(1-2-1) = u_1 + u_2 - 2u_1 - 3u_n - u_{n-1}$; i. e., $S_n = \frac{1}{2}[3u_n + u_{n-1} - 2]$.

Solved in a similar manner by J. Scheffer.

CALCULUS.

219. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

Evaluate (a) $\int_0^{\frac{1}{2}\pi} \frac{\sin mx \sin nx}{\sin x} dx$; (b) $\int_0^{\frac{1}{2}\pi} \frac{\cos mx \sin nx}{\sin x} dx$, where n is a positive integer. Also, modify the result for the case of m an integer.